

# Highest weight representations and Kac determinants for a class of conformal Galilei algebras with central extension

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## Abstract

We investigate the representations of a class of conformal Galilei algebras in one spatial dimension with central extension. This is done by explicitly constructing all singular vectors within the Verma modules, proving their completeness and then deducing irreducibility of the associated highest weight quotient modules. A resulting classification of infinite dimensional irreducible modules is presented. It is also shown that a formula for the Kac determinant is deduced from our construction of singular vectors. Thus we prove a conjecture of Dobrev, Doebner and Mrugalla for the case of the Schrödinger algebra.

## 1 Introduction

Conformal extensions of Galilei groups and their Lie algebras [1–3] are objects of physical and mathematical interest. In contrast to the relativistic conformal algebras there are some varieties of such algebras in a nonrelativistic setting even for a fixed dimension of spacetime. In this paper we investigate a class of conformal Galilei algebras specified by “spin”  $\ell$ . The member corresponding to the smallest value  $\ell = 1/2$  is the Schrödinger algebra [4, 5] whose relevance has been observed in a wide range of physical systems [4–32]. Other members of “spin- $\ell$ ” class algebra with  $\ell > \frac{1}{2}$  also play important roles in various fields in physics. For instance, one may encounter the algebra in classical mechanics with higher order time derivatives [33–35], electrodynamics [36], nonrelativistic analogue of AdS/CFT correspondence [37–41], nonrelativistic spacetime and gravity [42–44], quantum mechanical particle systems [45, 46], twistors [47] and so on. Furthermore, one may find applications of the algebraic structure to mathematical studies of topics such as systems of partial differential equation [48–50] and matrix orthogonal polynomials [51].

Despite the aforementioned fruitful physical applications, the representation theory of the “spin- $\ell$ ” conformal Galilei algebras has not been thoroughly developed. Presumably this is due to the non-semisimple nature of the algebra, namely, the algebra defined in  $d$ -dimensional space is a semi-direct sum of  $sl(2) \oplus so(d)$  (maximal semisimple subalgebra) and an abelian ideal depending on  $\ell$ . We remark that the abelian ideals for certain values of  $(d, \ell)$  have central extensions [11, 33, 34, 37]. One of the most fundamental problems in representation theory is a classification of irreducible representations. For the “spin- $\ell$ ” class of the conformal Galilei algebra this problem has been solved only in certain cases. Irreducible representations of lowest weight type for  $\ell = 1/2$  and  $d = 1, 2, 3$  are classified in [52, 53]. Two of the present authors gave the list of all possible irreducible representations

of highest weight type for  $\ell = 1$  and  $d = 2$  [54]. Although the classification problem of irreducible representations for other pairs of  $(d, \ell)$  is still open, these works show that the standard techniques for semisimple Lie algebras such as triangular decomposition, Verma modules and singular vectors can be applied to this particular class of non-semisimple Lie algebra. Motivated by this fact, in the current work we shall undertake further investigation of the classification problem. A goal of the present work is to give a list of all possible highest weight irreducible modules of the algebra with a central extension for  $d = 1$  and arbitrary half-integer  $\ell$ . This will be done by constructing singular vectors explicitly in Verma modules.

After the classification of irreducible modules, we derive the formula of the Kac determinant. Usually the Kac determinant is calculated in order to detect the existence of singular vectors. Here we are able to give the Kac determinant as a corollary of our explicit construction of singular vectors. In other words, we would like to show that our construction of singular vectors leads to an explicit formula for the Kac determinant. We include this because the calculation of the Kac determinant for the “spin- $\ell$ ” conformal Galilei algebra based on its definition is rather difficult. Even for the simplest example of  $\ell = 1/2$ ,  $d = 1$ , only a conjectured form of the Kac determinant has been given [52]. We give the Kac determinant for the  $d = 1$  algebra with arbitrary half-integer  $\ell$ . As a result we show that the conjectured formula presented in [52] is true.

We organize the paper as follows. In the next section we give the definition of  $d = 1$  algebra and fix our conventions. We then introduce the Verma modules and give the singular vectors in §3. It is shown that a Verma module has precisely one singular vector if it exists. Then follows a classification of irreducible modules of highest weight type. In §4 we derive the formula of the Kac determinant.

## 2 Structure of the conformal Galilei algebras, conventions and preliminaries

For<sup>1</sup>  $\ell \in \frac{1}{2}\mathbb{Z}^+$ , the “spin- $\ell$ ” class [2] of the conformal Galilei algebra (without central extension) defined in (1+1)-dimensional spacetime has a basis given by

$$\{C, D, H, P_n \mid n = 0, 1, 2, \dots, 2\ell\}.$$

Our purpose is to study the representation theory of the centrally extended algebra, which we denote  $\mathfrak{g}_\ell$ . It is well known [37] that in the case where  $\ell$  takes on integer values, no such central extension exists in the case of one spatial dimension. It is also known [52, 53] that the irreducible highest weight representations in the case  $\ell = 1/2$  algebra without central extension reduce to those of the  $sl(2)$  subalgebra spanned by  $C$ ,  $D$  and  $H$ . Therefore, in this article we focus only on the algebra with central extension, i.e.,  $\ell = 1/2, 3/2, 5/2, 7/2, \dots$ , i.e. the odd positive half-integers.

To this end, the non-zero defining relations of the algebra  $\mathfrak{g}_\ell$  under consideration are

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<sup>1</sup>Note that throughout the paper we use the following notation for the sets of non-negative and positive integers respectively:  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ ,  $\mathbb{Z}^+ = \{1, 2, \dots\}$ .

given by

$$\begin{aligned} [D, H] &= 2H, & [D, C] &= -2C, & [C, H] &= D, \\ [H, P_n] &= -nP_{n-1}, & [D, P_n] &= 2(\ell - n)P_n, & [C, P_n] &= (2\ell - n)P_{n+1}, \\ [P_m, P_n] &= I_{m,n}M. \end{aligned}$$

Here  $M$  is the central extension (so  $[M, a] = 0 \ \forall a \in \mathfrak{g}_\ell$ ), and the structure constants  $I_{m,n}$  form an antisymmetric tensor that can be determined by consistency with the Jacobi identity. In fact, from the Jacobi identity we first observe that the  $I_{m,n}$  can only be nontrivial if  $m + n = 2\ell$ , in which case the following linear equations must be satisfied:

$$(2\ell - n)I_{2\ell-n-1, n+1} = -(n+1)I_{2\ell-n, n}, \quad n = 0, 1, 2, \dots, 2\ell - 1.$$

Solving these equations gives

$$I_{m,n} = \delta_{m+n, 2\ell} (-1)^{m+\ell+\frac{1}{2}} m!n! \beta_\ell,$$

where  $\beta_\ell$  is an arbitrary constant that depends on  $\ell$ . For convenience we set

$$I_m = (-1)^{m+\ell+\frac{1}{2}} m! (2\ell - m)!, \quad (1)$$

and herein use the relation

$$[P_m, P_n] = \delta_{m+n, 2\ell} I_m M, \quad (2)$$

ignoring the factor  $\beta_\ell$  by absorbing it into the central extension  $M$ . We note that this agrees, up to an overall factor, with the structure constants used in [3] and [46].

One can immediately deduce a triangular decomposition of  $\mathfrak{g}_\ell$ :

$$\begin{aligned} \mathfrak{g}_\ell^+ &= \{ H, P_0, P_1, \dots, P_{\ell-\frac{1}{2}} \} \\ \mathfrak{g}_\ell^0 &= \{ D, M \} \\ \mathfrak{g}_\ell^- &= \{ C, P_{\ell+\frac{1}{2}}, P_{\ell+\frac{3}{2}}, \dots, P_{2\ell} \} \end{aligned}$$

We remark that  $\mathfrak{g}_\ell^\pm$  are Abelian for  $\ell = \frac{1}{2}$  and non-Abelian for  $\ell \geq \frac{3}{2}$ .

### 3 Singular vectors and irreducible modules

Our approach for finding all irreducible highest weight modules of  $\mathfrak{g}_\ell$  is according to the following procedure, which generalises that of [52] for the case  $\ell = 1/2$  (Schrödinger algebra).

1. Define a highest weight vector of the Verma module in a basis on which  $\mathfrak{g}_\ell^0$  is diagonal, and give the action of the generators in  $\mathfrak{g}_\ell$ , and hence a basis for the Verma module.
2. Determine a complete set of singular vectors in the Verma module.
3. Quotient out all highest weight submodules generated by the singular vectors. The remaining quotient module is then irreducible.

### 3.1 Verma module and basis

For fixed, real values of  $\delta$  and  $\mu$ , let  $|\delta, \mu\rangle$  be a highest weight vector in the Verma module, such that

$$D|\delta, \mu\rangle = \delta|\delta, \mu\rangle, \quad M|\delta, \mu\rangle = \mu|\delta, \mu\rangle, \quad X|\delta, \mu\rangle = 0, \quad X \in \mathfrak{g}_\ell^+.$$

The Verma module itself, denoted  $V^{\delta, \mu}$ , is determined by  $U(\mathfrak{g}_\ell^-)|\delta, \mu\rangle$ , with  $U(\mathfrak{g}_\ell^-)$  being the universal enveloping algebra of  $\mathfrak{g}_\ell^-$ . We then have a natural basis of  $V^{\delta, \mu}$  given by

$$\left\{ C^h \prod_{j=0}^{\ell-\frac{1}{2}} P_{\ell+\frac{1}{2}+j}^{k_j} |\delta, \mu\rangle \mid h, k_0, k_1, \dots, k_{\ell-\frac{1}{2}} \in \mathbb{Z}_+ \right\}.$$

It is easily seen that each basis vector is simultaneously an eigenvector of  $M$  and  $D$ . The eigenvalue of  $M$  is always  $\mu$ , since  $M$  is central. Using the commutation relations of  $D$  with powers of  $C$  and  $P_j$ , namely

$$[D, C^h] = -2hC^h, \quad [D, P_n^k] = 2k(n - \ell)P_n^k, \quad (3)$$

the eigenvalue of  $D$  corresponding to basis vector  $C^h \prod_{j=0}^{\ell-\frac{1}{2}} P_{\ell+\frac{1}{2}+j}^{k_j} |\delta, \mu\rangle$  is found to be

$$\delta - 2h - \sum_{j=0}^{\ell-\frac{1}{2}} (2j+1)k_j.$$

This suggests that we introduce the notion of *level* within  $V^{\delta, \mu}$ . We define the level  $m$  by

$$m = 2h + \sum_{j=0}^{\ell-\frac{1}{2}} (2j+1)k_j. \quad (4)$$

For a fixed value of  $m$  we take  $h, k_1, k_2, \dots, k_{\ell-\frac{1}{2}}$  as independent variables, since

$$k_0 = m - 2h - \sum_{j=1}^{\ell-\frac{1}{2}} (2j+1)k_j, \quad (5)$$

and in particular,

$$m - 2h - \sum_{j=1}^{\ell-\frac{1}{2}} (2j+1)k_j \geq 0. \quad (6)$$

We then find it convenient throughout the paper to denote the basis vectors at level  $m$  by  $|h, \underline{k}; m\rangle$ , where

$$|h, \underline{k}; m\rangle = C^h P_{\ell+\frac{1}{2}}^{m-2h-\sum_{j=1}^{\ell-\frac{1}{2}} (2j+1)k_j} \prod_{j=1}^{\ell-\frac{1}{2}} P_{\ell+\frac{1}{2}+j}^{k_j} |\delta, \mu\rangle, \quad (7)$$

with  $\underline{k} = (k_1, k_2, \dots, k_{\ell-\frac{1}{2}})$ . Note that the above expression appears cumbersome, so to facilitate readability of the equations throughout the paper, where possible we write  $k_0$  (which is always the power of the generator  $P_{\ell+\frac{1}{2}}$  occurring in the basis vectors) instead of the full expression in terms of  $m$ ,  $h$  and  $k_1, \dots, k_{\ell-\frac{1}{2}}$  as determined by equation (5). We then have expressions such as

$$|h, \underline{k}; m\rangle = C^h \prod_{j=0}^{\ell-\frac{1}{2}} P_{\ell+\frac{1}{2}+j}^{k_j} |\delta, \mu\rangle$$

making perfect sense by keeping equation (5) in mind.

The Verma module  $V^{\delta, \mu}$  is a graded module with respect to the vector space decomposition over  $m$ ,

$$V^{\delta, \mu} = \bigoplus_{m \in \mathbb{Z}_+} V_m^{\delta, \mu},$$

where  $V_m^{\delta, \mu}$  is the space spanned by the vectors  $|h, \underline{k}; m\rangle$  for fixed  $m$ .

### 3.2 Singular vectors

As mentioned in the outline at the start of this section, we proceed by stating an explicit form of singular vectors, and then show that these constitute a complete set of singular vectors in  $V^{\delta, \mu}$ . We remind the reader that the vector  $|\delta, \mu\rangle$  at level 0 is not considered to be a singular vector.

**Theorem 1** *If  $2\delta - 2(q-1) + (\ell + \frac{1}{2})^2 = 0$  for  $q \in \mathbb{Z}^+$  then the following is a singular vector at level  $2q$ :*

$$|u_{2q}\rangle = (\alpha_\ell \mu C - P_{\ell+\frac{1}{2}}^2)^q |\delta, \mu\rangle \in V_{2q}^{\delta, \mu} \quad (8)$$

where  $\alpha_\ell = 2((\ell - \frac{1}{2})!)^2$ .

**Proof:** Using the relations

$$\begin{aligned} [P_j, \alpha_\ell \mu C - P_{\ell+\frac{1}{2}}^2] &= -(2\ell - j) \alpha_\ell \mu P_{j+1}, \quad 0 \leq j \leq \ell - \frac{3}{2}, \\ [P_{\ell-\frac{1}{2}}, \alpha_\ell \mu C - P_{\ell+\frac{1}{2}}^2] &= \left(\ell + \frac{1}{2}\right) \alpha_\ell (M - \mu) P_{\ell+\frac{1}{2}}, \\ [H, \alpha_\ell \mu C - P_{\ell+\frac{1}{2}}^2] &= -\alpha_\ell \mu D + 2 \left(\ell + \frac{1}{2}\right) P_{\ell+\frac{1}{2}} P_{\ell-\frac{1}{2}} - \left(\left(\ell + \frac{1}{2}\right)!\right)^2 M, \end{aligned}$$

one may use a simple inductive argument on  $q$  to prove the following:

$$\begin{aligned} P_j |u_{2q}\rangle &= 0, \quad 0 \leq j \leq \ell - \frac{1}{2}, \\ H |u_{2q}\rangle &= -\frac{q}{2} \alpha_\ell \mu \left\{ 2\delta - 2(q-1) + \left(\ell + \frac{1}{2}\right)^2 \right\} |u_{2(q-1)}\rangle. \end{aligned}$$

It is straightforward to verify

$$D |u_{2q}\rangle = (\delta - 2q) |u_{2q}\rangle.$$

The result of the theorem follows. ■

Each singular vector given by Theorem 1 is a homogeneous vector in the Verma module. That is, for a given (even) level  $m$ , the singular vector  $|u_m\rangle$  can be expressed as a linear combination of the basis vectors of  $V_m^{\delta, \mu}$ :

$$|u_m\rangle = \sum_{h, \underline{k}} a(h, \underline{k}) |h, \underline{k}; m\rangle, \quad (9)$$

where the sum on  $h, \underline{k}$  is over all possible values satisfying (6).

We aim to show that the conditions  $X |u_m\rangle = 0$  for any  $X \in \mathfrak{g}_\ell^+$  determine all the coefficients  $a(h, \underline{k})$  uniquely up to overall constant. From this we are then able to deduce that the singular vectors given by Theorem 1 are the only singular vectors, and hence constitute a complete set.

We start with some useful preliminary results involving the defining relations, and remind the reader about the relation (2), and the definition of the structure constant  $I_m$  given in equation (1).

**Lemma 2** *The following commutation relations hold on  $U(\mathfrak{g}_\ell)$ :*

$$[P_j, C^h] = \sum_{n=1}^{\min\{h, 2\ell-j\}} (-1)^n n! \binom{h}{n} \binom{2\ell-j}{n} C^{h-n} P_{j+n}, \quad (10)$$

$$[P_j, P_{2\ell-j}^k] = k I_j M P_{2\ell-j}^{k-1}, \quad (11)$$

$$[H, C^h] = -h C^{h-1} D + h(h-1) C^{h-1}, \quad (12)$$

$$[H, P_n^k] = -k n P_n^{k-1} P_{n-1} + \delta_{n, \ell+\frac{1}{2}} \frac{1}{2} k(k-1) \left( \left( \ell + \frac{1}{2} \right)! \right)^2 M P_n^{k-2}. \quad (13)$$

**Proof:** Each relation can be proved using a straightforward inductive argument. The details are unenlightening and omitted. ■

Before proceeding, we introduce some useful notation to assist in keeping track of shifts in the state labels. Let  $\epsilon_j$  be the row vector in  $\mathbb{R}^{\ell-\frac{1}{2}}$  with 1 in the  $j$ -th entry and 0 elsewhere, i.e.,

$$\epsilon_j = (0, \dots, 0, 1, 0, \dots, 0).$$

We also use the notation

$$\underline{k}^{(r)} = (k_1, k_2, \dots, k_r, 0, \dots, 0) \in \mathbb{R}^{\ell-\frac{1}{2}}.$$

We adopt the convention that  $\epsilon_0 = (0, \dots, 0) = \underline{k}^{(0)}$ .

**Lemma 3** *The action of  $\mathfrak{g}_\ell^+ \cup \mathfrak{g}_\ell^0$  on  $|h, \underline{k}; m\rangle$  is given by the following expressions:*

$$M |h, \underline{k}; m\rangle = \mu |h, \underline{k}; m\rangle,$$

$$D|h, \underline{k}; m\rangle = (\delta - m)|h, \underline{k}; m\rangle, \quad (14)$$

$$\begin{aligned} P_{\ell-\frac{1}{2}-a}|h, \underline{k}; m\rangle &= \sum_{0 \leq n \leq a} (-1)^n k_{a-n} I_{\ell-\frac{1}{2}-a+n} n! \binom{h}{n} \binom{\ell + \frac{1}{2} + a}{n} \mu |h - n, \underline{k} - \underline{\epsilon}_{a-n}; m - 2a - 1\rangle \\ &+ \sum_{a+1 \leq n} (-1)^n n! \binom{h}{n} \binom{\ell + \frac{1}{2} + a}{n} |h - n, \underline{k} + \underline{\epsilon}_{n-a-1}; m - 2a - 1\rangle, \end{aligned} \quad (15)$$

where  $a = 0, 1, \dots, \ell - \frac{1}{2}$ .

$$\begin{aligned} H|h, \underline{k}; m\rangle &= h(m - h - \delta - 1)|h - 1, \underline{k}; m - 2\rangle \\ &+ \frac{1}{2} k_0(k_0 - 1) \left( \left( \ell + \frac{1}{2} \right)! \right)^2 \mu |h, \underline{k}; m - 2\rangle \\ &- \sum_{j=1}^{\ell-\frac{1}{2}} k_j \left( \ell + \frac{1}{2} + j \right) |h, \underline{k} + \underline{\epsilon}_{j-1} - \underline{\epsilon}_j; m - 2\rangle. \end{aligned} \quad (16)$$

**Proof:** Firstly, the action of  $M$  is trivial and only included for completeness. By application of the commutation relations (3) and those of Lemma 2, the remaining actions given in this Lemma can easily be deduced, so we omit details. ■

By the result of Lemma 3, in particular equations (15) and (16), one can then calculate the action of  $\mathfrak{g}_\ell^+$  on  $|u_m\rangle$ . We thus arrive at the following technical theorem which is crucial to our discussion.

**Theorem 4** *In order for the vector given in equation (9), (reproduced here)*

$$|u_m\rangle = \sum_{h, \underline{k}} a(h, \underline{k}) |h, \underline{k}; m\rangle,$$

*to be a singular vector,  $m$  must be even, in which case the coefficients  $a(h, \underline{k})$  are unique up to an overall factor.*

**Proof:** We determine conditions on  $a(h, \underline{k})$  such that  $X|u_m\rangle = 0, \forall X \in \mathfrak{g}_\ell^+$ .

We first note that the largest possible value of  $h$  for a given level  $m$  is  $\lfloor \frac{m}{2} \rfloor$ . More precisely, the following choice of  $h, k_0, \underline{k}$  gives the largest possible value of  $h$ :

$$\begin{aligned} m = 2q \text{ (even)} & \quad ; \quad h = q, \quad k_0 = 0, \quad \underline{k} = \underline{0}, \\ m = 2q + 1 \text{ (odd)} & \quad ; \quad h = q, \quad k_0 = 1, \quad \underline{k} = \underline{0}. \end{aligned}$$

By (15) with  $a = 0$  one has

$$\begin{aligned} P_{\ell-\frac{1}{2}}|u_m\rangle &= \sum_{h, \underline{k}} a(h, \underline{k}) \{ k_0 I_{\ell-\frac{1}{2}} \mu |h, \underline{k}; m - 1\rangle \\ &+ \sum_{1 \leq n} (-1)^n n! \binom{h}{n} \binom{h + \frac{1}{2}}{n} |h - n, \underline{k} + \underline{\epsilon}_{n-1}; m - 1\rangle \} \\ &= 0. \end{aligned} \quad (17)$$

We look at the coefficients of the vector  $|q, \varnothing; 2q - 1\rangle$  for even  $m$ , and  $|q, \varnothing; 2q\rangle$  for odd  $m$ . We see that there is no such vector for even  $m$  and only one for odd  $m$  with the coefficient  $a(q, \varnothing)I_{\ell-\frac{1}{2}}\mu$ . Thus we have

$$a(q, \varnothing) = \begin{cases} \text{arbitrary,} & \text{for } m = 2q, \\ 0, & \text{for } m = 2q + 1. \end{cases}$$

Next we look at the coefficients of  $|h, \varnothing; m - 1\rangle$  and obtain the recurrence relation:

$$(m - 2h)I_{\ell-\frac{1}{2}}\mu a(h, \varnothing) - (h + 1) \left( \ell + \frac{1}{2} \right) a(h + 1, \varnothing) = 0. \quad (18)$$

This recurrence relation is easily solved and we have

$$a(h, \varnothing) = \begin{cases} (-\alpha_\ell \mu)^{h-q} \binom{q}{h} a(q, \varnothing), & \text{for } m = 2q, \\ 0, & \text{for } m = 2q + 1. \end{cases}$$

This shows that  $a(h, \varnothing)$  is unique up to  $a(q, \varnothing)$ , for  $0 \leq h \leq q - 1$ .

Now we look at the coefficients of the vector  $|h, \underline{k}^{(r)}; m - 2\rangle$  in the equation  $H|u_m\rangle = 0$ . This is done by using (16) and gives the recurrence relation

$$\begin{aligned} & (h + 1)(m - h - 2 - \delta) a(h + 1, \underline{k}^{(r)}) + \frac{1}{2}k_0(k_0 - 1) \left( \left( \ell + \frac{1}{2} \right)! \right)^2 \mu a(h, \underline{k}^{(r)}) \\ & - \sum_{j=1}^r (k_j + 1) \left( \ell + \frac{1}{2} + j \right) a(h, \underline{k}^{(r)} - \epsilon_{j-1} + \epsilon_j) \\ & - \left( \ell + \frac{3}{2} + r \right) a(h, \underline{k}^{(r)} - \epsilon_r + \epsilon_{r+1}) = 0. \end{aligned} \quad (19)$$

This relation allows us to write  $a(h, \underline{k}^{(r)} + \epsilon_{r+1})$  as a linear combination of the coefficients  $a(s, \underline{k}^{(j)})$ , for all possible  $s$  and for  $j \leq r$ . For instance, setting  $r = 0$ , the relation (19) reads as follows

$$\begin{aligned} & (h + 1)(m - h - \delta - 2) a(h + 1, \varnothing) + \frac{1}{2}(m - 2h)(m - 2h - 1) \left( \left( \ell + \frac{1}{2} \right)! \right)^2 \mu a(h, \varnothing) \\ & - \left( \ell + \frac{3}{2} \right) a(h, \epsilon_1) = 0. \end{aligned} \quad (20)$$

We see that  $a(h, \epsilon_1)$  is therefore determined once we know coefficients of the form  $a(h, \varnothing)$  :

$$a(h, \epsilon_1) = \begin{cases} \text{unique up to } a(q, \varnothing), & \text{for } m = 2q, \\ 0, & \text{for } m = 2q + 1. \end{cases}$$

Next we look at coefficients of the vector  $|h - a, \underline{k}^{(a)}; m - 2a - 1\rangle$  in  $P_{\ell-\frac{1}{2}-a}|u_m\rangle = 0$ . This may be done by using (15). For  $0 \leq n \leq a - 1$  we make the replacement

$$h \rightarrow h + n - a, \quad k_{a-n} \rightarrow k_{a-n} + 1,$$



and for  $a + 1 \leq n \leq 2a + 1$ ,

$$h \rightarrow h + n - a, \quad k_{n-a-1} \rightarrow k_{n-a-1} - 1,$$

but no change for  $n = a$ . This leads to the following recurrence relation:

$$\begin{aligned} & \sum_{n=0}^{a-1} (-1)^n (k_{a-n} + 1) I_{\ell-\frac{1}{2}-a+n} n! \binom{h+n-a}{n} \left( \ell + \frac{1}{2} + a \right) \mu a(h+n-a, \tilde{k}^{(a)} + \xi_{a-n}) \\ & + (-1)^a k_0 I_{\ell-\frac{1}{2}} a! \binom{h}{a} \left( \ell + \frac{1}{2} + a \right) \mu a(h, \tilde{k}^{(a)}) \\ & + \sum_{n=a+1}^{2a+1} (-1)^n n! \binom{h+n-a}{n} \left( \ell + \frac{1}{2} + a \right) \mu a(h+n-a, \tilde{k}^{(a)} - \xi_{n-a-1}) = 0. \end{aligned} \quad (21)$$

The relation (21) relates the following coefficients:

$$\begin{aligned} & a(h-a, \tilde{k}^{(a)} + \xi_a), \quad a(h-a+1, \tilde{k}^{(a)} + \xi_{a-1}), \quad \dots, \quad a(h-1, \tilde{k}^{(a)} + \xi_1), \quad a(h, \tilde{k}^{(a)}), \\ & a(h+1, \tilde{k}^{(a)}), \quad a(h+2, \tilde{k}^{(a)} - \xi_1), \quad \dots, \quad a(h+a+1, \tilde{k}^{(a)} - \xi_a). \end{aligned}$$

Thus the relation (21) allows us to write  $a(h, \tilde{k}^{(a)} + \xi_a)$  as a linear combination of coefficients of the form  $a(s, \tilde{k}^{(a)})$  and  $a(s', \tilde{k}^{(a)} - \xi_j)$  for some  $s, s'$  and  $j \leq a$ . For instance, setting  $a = 1$  in (21) we have

$$\begin{aligned} & (k_1 + 1) I_{\ell-\frac{3}{2}} \mu a(h-1, \tilde{k}^{(1)} + \xi_1) - k_0 I_{\ell-\frac{1}{2}} h \left( \ell + \frac{3}{2} \right) \mu a(h, \tilde{k}^{(1)}) \\ & + \sum_{n=2}^3 (-1)^n n! \binom{h+n-1}{n} \left( \ell + \frac{3}{2} \right) \mu a(h+n-1, \tilde{k}^{(1)} - \xi_{n-2}) = 0. \end{aligned} \quad (22)$$

Repeated use of this relation leads to the conclusion

$$a(h, \tilde{k}^{(1)}) = \begin{cases} \text{unique up to } a(q, \emptyset), & \text{for } m = 2q, \\ 0, & \text{for } m = 2q + 1. \end{cases}$$

We remark that by setting  $k_1 = 0$ , the relation (22) gives the connection between  $a(h, \xi_1)$  and  $a(h, \emptyset)$ , in a similar way to (20):

$$I_{\ell-\frac{3}{2}} \mu a(h-1, \xi_1) - k_0 I_{\ell-\frac{1}{2}} h \left( \ell + \frac{3}{2} \right) \mu a(h, \emptyset) + 2 \binom{h+1}{2} \left( \ell + \frac{3}{2} \right) \mu a(h+1, \emptyset) = 0. \quad (23)$$

This also gives  $a(h, \xi_1)$  but it must be compatible with the previous computation from (20). For odd level  $m$  both (20) and (23) give  $a(h, \xi_1) = 0$  so that they are compatible. For even  $m = 2q$  with the aid of (18) it follows from (20) that

$$a(h, \xi_1) = \left( \ell - \frac{1}{2} \right) (q-h)(q-h-1) \alpha_\ell \mu a(h, \emptyset),$$

and from (23)

$$\left(\ell + \frac{3}{2}\right) a(h, \xi_1) = (q - h) \left\{ h + \delta - 2(q - 1) + \left(q - h - \frac{1}{2}\right) \left(\ell + \frac{1}{2}\right)^2 \right\} \alpha_{\ell\mu} a(h, \underline{0}).$$

Removing  $a(h, \xi_1)$  from the two above equations, one obtains the condition for  $\delta$  found in Theorem 1:

$$2\delta - 2(q - 1) + \left(\ell + \frac{1}{2}\right)^2 = 0. \quad (24)$$

We now return to the relation (19) and set  $r = 1$ . Then we see that

$$a(h, \underline{k}^{(1)} + \xi_2) = \begin{cases} \text{unique up to } a(q, \underline{0}), & \text{for } m = 2q, \\ 0, & \text{for } m = 2q + 1. \end{cases}$$

By the relation (21) with  $a = 2$ , it is not difficult to see

$$a(h, \underline{k}^{(2)}) = \begin{cases} \text{unique up to } a(q, \underline{0}), & \text{for } m = 2q, \\ 0, & \text{for } m = 2q + 1. \end{cases}$$

Repeating this it may be proved that

$$a(h, \underline{k}) = \begin{cases} \text{unique up to } a(q, \underline{0}), & \text{for } m = 2q, \\ 0, & \text{for } m = 2q + 1, \end{cases}$$

which is enough to complete the proof of the theorem. ■

We have shown that  $a(h, \underline{k}) = 0$  for odd  $m$  and uniquely determined up to  $a(q, \underline{0})$  for even  $m$ . Therefore, only at even levels  $m = 2q$  do singular vectors exist in  $V^{\delta, \mu}$ , with  $\delta$  satisfying (24). Hence we may proceed with a classification of irreducible highest weight modules of  $\mathfrak{g}_\ell$ .

### 3.3 Classification of irreducible highest weight modules

We have shown that  $V^{\delta, \mu}$  with  $\delta$  satisfying (24) has precisely one singular vector given by (8). It follows that  $V^{\delta, \mu}$  contains the invariant submodule

$$I^{\delta, \mu} = U(\mathfrak{g}_\ell^-) |u_{2q}\rangle.$$

Now we show that there are no singular vectors in the quotient module  $V^{\delta, \mu}/I^{\delta, \mu}$  so that the quotient module is irreducible. Let  $|0\rangle$  be the highest weight vector in  $V^{\delta, \mu}/I^{\delta, \mu}$ . A basis of  $V^{\delta, \mu}/I^{\delta, \mu}$  is then given by

$$\left\{ C^h \prod_{j=0}^{\ell-\frac{1}{2}} P_{\ell+\frac{1}{2}+j}^{k_j} |0\rangle \mid h = 0, 1, \dots, q-1, k_j \in \mathbb{Z}_+ \right\}.$$

For a fixed level  $m$  we denote a basis vector by  $|h, \underline{k}; m\rangle$  and a singular vector by

$$|u_m\rangle = \sum_{h, \underline{k}} a(h, \underline{k}) |h, \underline{k}; m\rangle,$$

where the sum over  $h$  is restricted to  $0 \leq h \leq q-1$ . The condition  $P_{\ell-\frac{1}{2}}|u_m\rangle = 0$  has the same form as (17). We study the coefficient  $a(h, \underline{0})$  with the largest possible value of  $h$ .

(i)  $m = 2p + 1$  (odd)

- $p > q - 1$   
 $(h, k_0, \underline{k}) = (q-1, 2(p-q)+3, \underline{0})$  corresponds to the largest possible choice of  $h$ . The corresponding vector is  $|q-1, \underline{0}; 2p\rangle$  with the coefficients  $a(q-1, \underline{0})k_0I_{\ell-\frac{1}{2}}\mu$ . Therefore  $a(q-1, \underline{0}) = 0$ .
- $p \leq q - 1$   
 $(h, k_0, \underline{k}) = (p, 1, \underline{0})$  corresponds to the largest possible choice of  $h$ . The corresponding term is  $a(p, \underline{0})k_0I_{\ell-\frac{1}{2}}\mu|p, \underline{0}; 2p\rangle$  so that  $a(p, \underline{0}) = 0$ .

By the same argument as the proof of Theorem 4 one may show that  $a(h, \underline{k}) = 0$ . Therefore there are no singular vectors in  $I^{\delta, \mu}$  at odd levels.

(ii)  $m = 2p$  (even)

- $p > q - 1$   
 $(h, k_0, \underline{k}) = (q-1, 2(p-q+1), \underline{0})$  corresponds to the largest possible choice of  $h$ . The corresponding vector  $|q-1, \underline{0}; 2p-1\rangle$  has nonvanishing coefficient so that  $a(q-1, \underline{0}) = 0$ .
- $p \leq q - 1$   
 $(h, k_0, \underline{k}) = (p, 0, \underline{0})$  corresponds to the largest possible choice of  $h$ . The corresponding vector has vanishing coefficient ( $k_0 = 0$ ) so that  $a(p, \underline{0})$  remains undetermined. However, by a similar argument to the proof of Theorem 4, one requires

$$2\delta - 2(p-1) + \left(\ell + \frac{1}{2}\right)^2 = 0, \quad (25)$$

as the consistency condition. This condition is never satisfied since  $p \leq q-1$  and (24) is supposed. This means that one may not determine  $a(h, \underline{k})$ .

Therefore there are no singular vectors in  $I^{\delta, \mu}$  at even levels.

We thus arrive at the following theorem.

**Theorem 5** *The irreducible highest weight modules of  $\mathfrak{g}_\ell$  for odd half-integer  $\ell$  with non-vanishing  $\mu$  are listed as follows:*

- $V^{\delta, \mu}$  if  $\delta \neq q-1 - \frac{1}{2}\left(\ell + \frac{1}{2}\right)^2$ ,
- $V^{\delta, \mu}/I^{\delta, \mu}$  if  $\delta = q-1 - \frac{1}{2}\left(\ell + \frac{1}{2}\right)^2$ ,

where  $q \in \mathbb{Z}^+$ . All modules are infinite dimensional.

## 4 Kac determinant

Although we have already deduced a characterisation of the irreducible highest weight modules of  $\mathfrak{g}_\ell$  as presented in Theorem 5, we find that we are able to also give the form of the Kac determinant corresponding to the subspace  $V_m^{\delta,\mu}$  of the Verma module at arbitrary level  $m$ .

We define a sesquilinear form  $(,)$  on  $V_m^{\delta,\mu}$  (Shapovalov form [55]) by setting

$$\langle \delta, \mu | \delta, \mu \rangle \equiv (|\delta, \mu\rangle, |\delta, \mu\rangle) = 1,$$

and

$$(A|x\rangle, B|y\rangle) = (|x\rangle, \omega(A)B|y\rangle), \quad \forall |x\rangle, |y\rangle \in V_m^{\delta,\mu}, \quad A, B \in \mathfrak{g}_\ell,$$

where  $\omega$  is an algebra anti-automorphism defined by

$$\omega(P_j) = P_{2\ell-j}, \quad \omega(C) = H, \quad \omega(H) = C, \quad \omega(D) = D, \quad \omega(M) = M.$$

Note that  $\omega$  is involutive, i.e. satisfies  $\omega^2 = \text{id}$ . Generally, the form  $(,)$  is hermitian. Restricting  $(,)$  to the basis at level  $m$  determined by the vectors (7), we have the following.

**Lemma 6** *The form  $(,)$  is symmetric on the basis  $\{|h, \underline{k}; m\rangle\}$  of  $V_m^{\delta,\mu}$ .*

**Proof:** Since  $\omega^2 = \text{id}$ , we clearly have  $(\omega(A)x, y) = (x, \omega^2(A)y) = (x, Ay)$ . Let  $A|\delta, \mu\rangle$  and  $B|\delta, \mu\rangle$  be two vectors in the basis  $\{|h, \underline{k}; m\rangle\}$  of  $V_m^{\delta,\mu}$ . Then

$$\begin{aligned} (A|\delta, \mu\rangle, B|\delta, \mu\rangle) &= (|\delta, \mu\rangle, \omega(A)B|\delta, \mu\rangle) \\ &= \begin{cases} \alpha; & \text{if } \omega(A)B|\delta, \mu\rangle = \alpha|\delta, \mu\rangle \\ 0; & \text{otherwise} \end{cases} \\ &= (\omega(A)B|\delta, \mu\rangle, |\delta, \mu\rangle) = (B|\delta, \mu\rangle, A|\delta, \mu\rangle). \end{aligned}$$

Note that  $\alpha \in \mathbb{R}$  in the above calculation. ■

Given an ordered basis  $\{v_i\}$  of the level  $m$  subspace  $V_m^{\delta,\mu}$  of  $V^{\delta,\mu}$ , we define a matrix whose entry in the  $i$ th row and  $j$ th column is the number  $(v_i, v_j)$ . Clearly the null space of this matrix will lead to the set of vectors in  $V_m^{\delta,\mu}$ , called *null vectors*, that are orthogonal to vectors in the basis. Elementary linear algebra then tells us that null vectors only exist if and only if the determinant of the matrix is zero. This determinant is called the *Kac determinant at level  $m$* . In what follows, for  $|x\rangle \in V^{\delta,\mu}$ , we call  $A|x\rangle$  a *descendent* of  $|x\rangle$  for any  $A \in \mathfrak{g}_\ell^-$ . The following lemma is included as a summary of some well-known results (e.g. see [56]) concerning singular vectors and descendents of null vectors.

**Lemma 7** *A singular vector is also a null vector, and descendents of a null vector are also null vectors.*

**Proof:** Let  $v_s$  be a singular vector at level  $m$ . Note that any vector  $x$  at level  $m$  can be written as  $x = Ax'$ , for some  $A \in \mathfrak{g}_\ell^-$  and  $x'$  a vector whose level is less than  $m$ . Note then that  $\omega(A) \in \mathfrak{g}_\ell^+$ , and therefore  $(x, v_s) = (x', \omega(A)v_s) = 0$ . Hence  $v_s$  is null. By a similar

argument, if  $v$  is null, then  $(x, Bv) = (\omega(B)x, v) = 0$ , and hence  $Bv$  is also null for all  $B \in \mathfrak{g}_\ell^-$ . ■

The problem of determining singular vectors and finding highest weight submodules in the Verma module is often associated with calculating the Kac determinant (e.g. see [56]). Indeed, the Kac determinant is usually used as a tool to undertake such analysis. Using results of the previous sections, especially the completeness results of the singular vectors (Theorem 4), we find that we can actually deduce a formula for the Kac determinant. We include this in our paper more as a curious corollary to our main results, rather than as a practical tool.

#### 4.1 Dimension of $V_m^{\delta, \mu}$

Before looking at the details of the Kac determinant, we remark on the dimension of the level subspace  $V_m^{\delta, \mu}$ . We note that the vectors in the basis  $\{|h, k; m\rangle\}$  given in (7) for fixed level  $m$  are in one to one correspondence with the restricted set of integer partitions of the integer  $m$ , with parts taken from the subset of integers

$$\{2\} \cup \left\{ 2j+1 \mid j = 0, 1, \dots, \ell - \frac{1}{2} \right\}. \quad (26)$$

Observe that the vectors (7) in the basis of  $V_m^{\delta, \mu}$  can be enumerated by the nested labelling

$$\left\{ (h, k_1, \dots, k_j, \dots, k_{\ell-\frac{1}{2}}) \mid 0 \leq h \leq \left\lfloor \frac{m}{2} \right\rfloor, 0 \leq k_{\ell-\frac{1}{2}} \leq \left\lfloor \frac{m-2h}{2(\ell-\frac{1}{2})+1} \right\rfloor, \dots \right. \\ \left. \dots 0 \leq k_j \leq \left\lfloor \frac{m-2h - \sum_{n=j+1}^{\ell-\frac{1}{2}} (2n+1)k_n}{2j+1} \right\rfloor, \dots, \right. \\ \left. \dots 0 \leq k_1 \leq \left\lfloor \frac{m-2h - \sum_{n=2}^{\ell-\frac{1}{2}} (2n+1)k_n}{3} \right\rfloor \right\}. \quad (27)$$

We therefore obtain the awkward yet explicit formula for the dimension of  $V_m^{\delta, \mu}$ , denoted  $d_m^\ell$ :

$$d_m^\ell = \sum_{h=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{k_{\ell-\frac{1}{2}}=0}^{\left\lfloor \frac{m-2h}{2(\ell-\frac{1}{2})+1} \right\rfloor} \dots \sum_{k_j=0}^{\left\lfloor \frac{m-2h - \sum_{n=j+1}^{\ell-\frac{1}{2}} (2n+1)k_n}{2j+1} \right\rfloor} \dots \sum_{k_1=0}^{\left\lfloor \frac{m-2h - \sum_{n=2}^{\ell-\frac{1}{2}} (2n+1)k_n}{3} \right\rfloor} 1.$$

For example, the first few cases of  $\ell$  gives

$$\begin{aligned}
d_m^{1/2} &= \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} 1, \\
d_m^{3/2} &= \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k_1=0}^{\lfloor \frac{m-2h}{3} \rfloor} 1, \\
d_m^{5/2} &= \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{m-2h}{5} \rfloor} \sum_{k_1=0}^{\lfloor \frac{m-2h-5k_2}{3} \rfloor} 1, \\
d_m^{7/2} &= \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k_3=0}^{\lfloor \frac{m-2h}{7} \rfloor} \sum_{k_2=0}^{\lfloor \frac{m-2h-7k_3}{5} \rfloor} \sum_{k_1=0}^{\lfloor \frac{m-2h-7k_3-5k_2}{3} \rfloor} 1.
\end{aligned}$$

A more elegant approach is to characterise  $d_m^\ell$  by the generating function [57, 58]

$$F^\ell(x) = \frac{1}{1-x^2} \prod_{j=0}^{\ell-\frac{1}{2}} \frac{1}{1-x^{2j+1}}, \quad (28)$$

in the sense that the coefficients of the formal power series are the numbers  $d_m^\ell$ , i.e.

$$F^\ell(x) = \sum_{m=0}^{\infty} d_m^\ell x^m.$$

For example, using this generating function, we can apply standard combinatorial techniques [57] to determine explicit formulas for the first few values of  $\ell$ :

$$\begin{aligned}
d_m^{1/2} &= \left\lfloor \frac{m+2}{2} \right\rfloor, \\
d_m^{3/2} &= \left\lfloor \frac{m^2 + 6m + 12}{12} \right\rfloor, \\
d_m^{5/2} &= \left\lfloor \frac{2m^3 + 33m^2 + 162m + 360}{360} \right\rfloor, \\
d_m^{7/2} &= \left\lfloor \frac{m^4 + 36m^3 + 442m^2 + 2124m + 5040}{5040} \right\rfloor.
\end{aligned} \quad (29)$$

Here we have used the notation  $\lfloor x \rfloor = \max\{N \in \mathbb{Z} \mid N \leq x\}$  is the usual floor function.

## 4.2 Dependence on $\delta$ in the Kac determinant

The result of Theorem 1 establishes the existence of singular vectors for certain values of  $\delta$ . Since all singular vectors are null vectors by Lemma 7, the Kac determinant must contain factors of the form  $2\delta - 2(q-1) + (\ell + \frac{1}{2})^2$ , arising from the existence condition for

a singular vector at level  $2q$ . For the Kac determinant at level  $m$ , such a factor must have algebraic multiplicity that is greater than or equal to the number of linearly independent descendents of the singular vector at level  $2q \leq m$ . We therefore have a lower bound on the algebraic multiplicity.

**Lemma 8** *The Kac determinant at level  $m$  contains the factor*

$$\left(2\delta - 2(q-1) + \left(\ell + \frac{1}{2}\right)^2\right)^{d_{m-2q}^\ell},$$

for every integer  $q > 0$  satisfying  $m \geq 2q$ .

Further to this, we can also see that only certain entries of the matrix of Shapovalov forms involve factors of  $\delta$ , namely those entries whose Shapovalov form contains  $C$  generators in *both* basis vectors (e.g.  $(C^2 P_{\ell+\frac{1}{2}}^2 |\delta, \mu\rangle, C^3 |\delta, \mu\rangle)$  will be quadratic in  $\delta$ ). In the following, we use  $a \sim \delta^h$  to indicate that  $a$  is a polynomial of degree  $h$  in  $\delta$ .

**Lemma 9** *Without loss of generality, let  $h \leq h'$ . Then  $\langle h, \underline{k}; m | h', \underline{k}'; m \rangle$  is either zero or*

$$\langle h, \underline{k}; m | h', \underline{k}'; m \rangle \sim \delta^h$$

**Proof:** It is a trivial matter to verify that some Shapovalov forms are zero. In the case it is non-zero, we employ the commutations relations, particularly

$$[H, C^{h'}] = -h' C^{h'-1} D + h'(h' - 1) C^{h'-1}, \quad [D, P_n^k] = 2k(\ell - n) P_n^k.$$

We then have

$$\begin{aligned} (C^h \mathbb{P} |\delta, \mu\rangle, C^{h'} \mathbb{P}' |\delta, \mu\rangle) &= (C^{h-1} \mathbb{P} |\delta, \mu\rangle, H C^{h'} \mathbb{P}' |\delta, \mu\rangle), \\ &= -h'(C^{h-1} \mathbb{P} |\delta, \mu\rangle, C^{h'-1} D \mathbb{P}' |\delta, \mu\rangle) \\ &\quad + h'(h' - 1)(C^{h-1} \mathbb{P} |\delta, \mu\rangle, C^{h'-1} \mathbb{P}' |\delta, \mu\rangle), \end{aligned}$$

where  $\mathbb{P}$  and  $\mathbb{P}'$  represent some unimportant product of the  $P_n$  generators. We arrive at the result by straightforward induction on  $h$ . Note also that we lose no generality by setting  $h \leq h'$  because  $(,)$  is symmetric on the basis according to Lemma 6. ■

The significance of Lemma 9 is that the diagonal entries of the matrix will contain polynomials in  $\delta$  of (non-strict) maximal degree in each row. Therefore, the degree of the polynomial in  $\delta$  occurring in the Kac determinant will be the sum of the degrees in the diagonal entries, which in turn is just the number of  $C$  generators occurring in all basis vectors of  $V_m^{\delta, \mu}$ .

In the following Lemma, we denote by  $O_n^{2\ell}$  the number of integer partitions of  $n$  comprising only odd parts no greater than  $2\ell$ . For convenience we adopt the convention that  $O_0^{2\ell} = 1$ .

**Lemma 10**

$$d_m^\ell = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} O_{m-2n}^{2\ell}$$

**Proof:** It is well known (e.g. see [57, 58]) that a generating function for the number of integer partitions of  $n$  comprising only odd parts no greater than  $2\ell$  is given by

$$\prod_{j=0}^{\ell-\frac{1}{2}} \frac{1}{1-x^{2j+1}} = \sum_{n=0}^{\infty} O_n^{2\ell} x^n,$$

which occurs in the generating function (28) for the dimensions of the level subspaces  $V_m^{\delta, \mu}$ . Indeed, we see immediately that

$$\begin{aligned} \sum_{m=0}^{\infty} d_m^{\ell} x^m = F^{\ell}(x) &= \frac{1}{1-x^2} \prod_{j=0}^{\ell-\frac{1}{2}} \frac{1}{1-x^{2j+1}} \\ &= \sum_{n=0}^{\infty} x^{2n} \sum_{t=0}^{\infty} O_t^{2\ell} x^t \\ &= \sum_{n,t=0}^{\infty} O_t^{2\ell} x^{2n+t} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} O_{m-2n}^{2\ell} x^m \text{ (setting } m = 2n + t) \end{aligned}$$

from which the result follows. ■

It is clear that the number of vectors in the basis of  $V_m^{\delta, \mu}$  containing  $C^h$  must be  $O_{m-2h}^{2\ell}$ . It follows that the number of times the  $C$  generators appear in the basis vectors of  $V_m^{\delta, \mu}$  is given by the expression

$$\sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} n O_{m-2n}^{2\ell}.$$

As mentioned in the discussion following Lemma 9, this is precisely the degree of the polynomial in  $\delta$  that occurs in the Kac determinant. The result of Lemma 10 then implies that

$$\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} d_{m-2(j+1)}^{\ell} = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} n O_{m-2n}^{2\ell},$$

obtained by a simple rearrangement of the summation on the right hand side of the equation in Lemma 10. The left hand side coincides with the sum of the powers of the factors obtained in Lemma 8. Therefore the result of Lemma 8 is not just a lower bound on the algebraic multiplicity, it is precisely the algebraic multiplicity. We now have all the pieces required to state the following.

**Theorem 11** *The Kac determinant at level  $m$ , denoted  $\mathcal{D}_m^{\ell}$ , is of the form*

$$\mathcal{D}_m^{\ell} = f(\mu) \prod_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} \left( 2\delta - 2j + \left( \ell + \frac{1}{2} \right)^2 \right)^{d_{m-2(j+1)}^{\ell}}.$$



Note that the function  $f(\mu)$  remains as yet undetermined. At this stage we only make the obvious point that the Kac determinant must depend on  $\mu$ . We now turn to determining the form of  $f(\mu)$ .

### 4.3 Dependence on $\mu$ in the Kac determinant

It is convenient to introduce the notion of  $\mu$ -weight of a basis vector. Let the basis  $\{|h, \underline{k}; m\rangle\}$  of  $V_m^{\delta, \mu}$  be denoted by  $\gamma$ . For any  $v \equiv |h, \underline{k}; m\rangle \in \gamma$  we define the  $\mu$ -weight of  $v$ , denoted  $\rho_v$ , as

$$\rho_v = m - 2 \left( h + \sum_{j=1}^{\ell - \frac{1}{2}} j k_j \right). \quad (30)$$

Note that the  $\mu$ -weight of a vector is nothing more than the sum of powers of all the  $P_n$ -type generators appearing in that basis vector. For example, in the case  $\ell = \frac{5}{2}$ , consider the following three basis vectors at level 8:

$$\begin{aligned} u &\equiv |1, \underline{0}; 8\rangle = CP_3^6 |\delta, \mu\rangle, \\ v &\equiv |1, \underline{\epsilon}_1; 8\rangle = CP_3^3 P_4 |\delta, \mu\rangle, \\ w &\equiv |0, \underline{\epsilon}_1 + \underline{\epsilon}_2; 8\rangle = P_4 P_5 |\delta, \mu\rangle. \end{aligned}$$

Their  $\mu$ -weights are given by

$$\rho_u = 6, \quad \rho_v = 4, \quad \rho_w = 2.$$

**Lemma 12** *For all  $v, w \in \gamma$ , we have that either  $(v, w) = 0$  or*

$$(v, w) = Z \mu^{\frac{1}{2}(\rho_v + \rho_w)},$$

*for some  $Z$  that has no dependence on  $\mu$ .*

**Proof:** As in the case of Lemma 9, it is easy to see that the Shapovalov forms between some pairs of basis vectors are zero. In the case the form is non-zero, an inductive proof by level can be used to prove the result. We outline such a calculation here. Firstly the result is true at level 0 since  $\langle \delta, \mu | \delta, \mu \rangle = 1$ . We now look at the form

$$\langle h, \underline{k}; m | h', \underline{k}'; m \rangle,$$

insisting only that  $h \neq 0$ , and assume the statement is true for all levels lower than this one. Using the results of Lemma 2, we have

$$\begin{aligned} \langle h, \underline{k}; m | h', \underline{k}'; m \rangle &= - \sum_{n=1}^{\ell - \frac{1}{2}} k'_n \langle h - 1, \underline{k}; m - 2 | h', k' + \underline{\epsilon}_{n-1} - \underline{\epsilon}_n; m - 2 \rangle \\ &\quad + \frac{1}{2} k'_0 (k'_0 - 1) \left( \left( \ell + \frac{1}{2} \right)! \right)^2 \mu \langle h - 1, \underline{k}; m - 2 | h', \underline{k}'; m - 2 \rangle \\ &\quad + h' (m - h' - 1 - \delta) \langle h - 1, \underline{k}; m - 2 | h' - 1, \underline{k}'; m - 2 \rangle. \end{aligned}$$

By the inductive assumption, we have

$$\begin{aligned}
\langle h-1, \underline{k}; m-2 | h', \underline{k}' + \underline{\epsilon}_{n-1} - \underline{\epsilon}_n; m-2 \rangle &= Z_1 \mu^{\frac{1}{2} \left( m-2-2 \left( h-1 + \sum_{j=1}^{\ell-\frac{1}{2}} j k_j \right) + m-2-2 \left( h' + \sum_{j=1}^{\ell-\frac{1}{2}} j k'_j + (n-1)-n \right) \right)} \\
&= Z_1 \mu^{\frac{1}{2} \left( m-2 \left( h + \sum_{j=1}^{\ell-\frac{1}{2}} j k_j \right) + m-2 \left( h' + \sum_{j=1}^{\ell-\frac{1}{2}} j k'_j \right) \right)}, \\
\langle h-1, \underline{k}; m-2 | h', \underline{k}'; m-2 \rangle &= Z_2 \mu^{\frac{1}{2} \left( m-2-2 \left( h-1 + \sum_{j=1}^{\ell-\frac{1}{2}} j k_j \right) + m-2 \left( h' + \sum_{j=1}^{\ell-\frac{1}{2}} j k'_j \right) \right)} \\
&= Z_2 \mu^{-1+\frac{1}{2} \left( m-2 \left( h + \sum_{j=1}^{\ell-\frac{1}{2}} j k_j \right) + m-2 \left( h' + \sum_{j=1}^{\ell-\frac{1}{2}} j k'_j \right) \right)}, \\
\langle h-1, \underline{k}; m-2 | h'-1, \underline{k}'; m-2 \rangle &= Z_3 \mu^{\frac{1}{2} \left( m-2-2 \left( h-1 + \sum_{j=1}^{\ell-\frac{1}{2}} j k_j \right) + m-2-2 \left( h'-1 + \sum_{j=1}^{\ell-\frac{1}{2}} j k'_j \right) \right)} \\
&= Z_3 \mu^{\frac{1}{2} \left( m-2 \left( h + \sum_{j=1}^{\ell-\frac{1}{2}} j k_j \right) + m-2 \left( h' + \sum_{j=1}^{\ell-\frac{1}{2}} j k'_j \right) \right)},
\end{aligned}$$

and so it is clear that in this case

$$\langle h, \underline{k}; m | h', \underline{k}'; m \rangle = Z \mu^{\frac{1}{2} \left( m-2 \left( h + \sum_{j=1}^{\ell-\frac{1}{2}} j k_j \right) + m-2 \left( h' + \sum_{j=1}^{\ell-\frac{1}{2}} j k'_j \right) \right)},$$

as required, using the definition of  $\mu$ -weights. Note that in the above calculation,  $Z_1$ ,  $Z_2$ ,  $Z_3$  and  $Z$  are unimportant expressions with no  $\mu$  dependence. We now consider  $\langle 0, \underline{k}_{(n)}; m | 0, \underline{k}'; m \rangle$ , where the notation  $\underline{k}_{(n)}$  implies that we impose the constraint  $k_0 = k_1 = \dots = k_{n-1} = 0$ , and we apply this for each permissible  $0 \leq n \leq \ell - \frac{1}{2}$ . In other words, we consider products of the form

$$\left( \prod_{j=n}^{\ell-\frac{1}{2}} P_{\ell+\frac{1}{2}+j}^{k_j} | \delta, \mu \rangle, \prod_{j=0}^{\ell-\frac{1}{2}} P_{\ell+\frac{1}{2}+j}^{k'_j} | \delta, \mu \rangle \right),$$

such that

$$\sum_{j=n}^{\ell-\frac{1}{2}} (2j+1) k_j = m = \sum_{j=0}^{\ell-\frac{1}{2}} (2j+1) k'_j.$$

Again, using the results of Lemma 2, it is not difficult to verify that

$$\langle 0, \underline{k}_{(n)}; m | 0, \underline{k}'; m \rangle = k'_n I_{\ell-\frac{1}{2}-n} \mu \langle 0, \underline{k}_{(n)} - \underline{\epsilon}_n; m - (2n+1) | 0, \underline{k}' - \underline{\epsilon}_n; m - (2n+1) \rangle.$$

By the inductive assumption, we have

$$\begin{aligned}
&\langle 0, \underline{k}_{(n)} - \underline{\epsilon}_n; m - (2n+1) | 0, \underline{k}' - \underline{\epsilon}_n; m - (2n+1) \rangle \\
&= Z' \mu^{\frac{1}{2} \left( m-(2n+1)-2 \left( \sum_{j=n}^{\ell-\frac{1}{2}} j k_j - n \right) + m-(2n+1)-2 \left( \sum_{j=0}^{\ell-\frac{1}{2}} j k'_j - n \right) \right)} \\
&= Z' \mu^{-1+\frac{1}{2} \left( m-2 \left( \sum_{j=n}^{\ell-\frac{1}{2}} j k_j \right) + m-2 \left( \sum_{j=0}^{\ell-\frac{1}{2}} j k'_j \right) \right)}
\end{aligned}$$

and hence

$$\langle 0, \underline{k}_{(n)}; m | 0, \underline{k}'; m \rangle = Z'' \mu^{\frac{1}{2} \left( m-2 \left( \sum_{j=n}^{\ell-\frac{1}{2}} j k_j \right) + m-2 \left( \sum_{j=0}^{\ell-\frac{1}{2}} j k'_j \right) \right)}$$

as required, with  $Z'$  and  $Z''$  being unimportant expressions with no  $\mu$  dependence. The result is thus proved by induction, as we have now covered all possibilities of the Shapovalov form using the fact that  $\langle \cdot | \cdot \rangle$  is symmetric on the basis, i.e. the result of Lemma 6.  $\blacksquare$

Now that we have established how  $\mu$  occurs in the Shapovalov forms, we are able to deduce that  $\mu$  occurs as a monomial in the Kac determinant. The simplest way of viewing this is to recall the Leibniz formula for the determinant of a matrix. For the purpose of readability, we set the size of the matrix to be  $n$  ( $= d_m^\ell$ ). Representing the basis vectors of  $V_m^{\delta, \mu}$  as  $\{v_i \mid i = 1, 2, \dots, n\}$ , and recalling the Levi-Civita antisymmetric tensor

$$\epsilon^{i_1 i_2 \dots i_n} = \begin{cases} 1; & (i_1, i_2, \dots, i_n) \text{ even permutation of } (1, 2, \dots, n) \\ -1; & (i_1, i_2, \dots, i_n) \text{ odd permutation of } (1, 2, \dots, n) \\ 0; & \text{otherwise,} \end{cases}$$

the Kac determinant can be expressed in the form

$$\mathcal{D}_m^\ell = \sum_{i_1, i_2, \dots, i_n=1}^n \epsilon^{i_1 i_2 \dots i_n} (v_1, v_{i_1}) (v_2, v_{i_2}) \cdots (v_n, v_{i_n}).$$

The result of Lemma 12 then implies that

$$\mathcal{D}_m^\ell = \sum_{i_1, i_2, \dots, i_n=1}^n Z_{i_1 i_2 \dots i_n} \epsilon^{i_1 i_2 \dots i_n} \mu^{\frac{1}{2}(\rho_{v_1} + \rho_{v_{i_1}})} \mu^{\frac{1}{2}(\rho_{v_2} + \rho_{v_{i_2}})} \cdots \mu^{\frac{1}{2}(\rho_{v_n} + \rho_{v_{i_n}})},$$

where the  $Z_{i_1 i_2 \dots i_n}$  are expressions that have no dependence on  $\mu$ . Since  $\{i_1, i_2, \dots, i_n\}$  label all of the basis vectors, we must have

$$\mathcal{D}_m^\ell = Z \mu^{\sum_{i=1}^n \rho_{v_i}}.$$

The factor  $Z$  has no dependence on  $\mu$ , but it may have dependence on  $\delta$  depending on the level  $m$ . This dependence on  $\delta$  in the Kac determinant has already been described in Theorem 11.

Rather than give the final formula for the Kac determinant in terms of  $\mu$ -weights, which depends on knowledge of the basis, we seek a final expression in terms of  $\ell$  and  $m$  only. We denote by  $e_m^\ell$  the sum of the  $\mu$ -weights over the basis, i.e.

$$e_m^\ell = \sum_{v \in \gamma} \rho_v, \quad (31)$$

where  $\gamma$  represents the basis  $\{|h, \underline{k}; m\rangle\}$  of  $V_m^{\delta, \mu}$ . In other words,  $e_m^\ell$  is the total number of  $P_n$ -type generators that occur in the basis  $\gamma$ , which is furthermore equivalent to the total number of odd parts occurring in the restricted integer partitions with parts taken from the subset of integers (26). We can use the basis labelling given in the expression (27) along with the definition of  $\mu$ -weight given in equation (30) to rewrite equation (31) as

$$e_m^\ell = \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k_{\ell-\frac{1}{2}}=0}^{\lfloor \frac{m-2h}{2(\ell-\frac{1}{2})+1} \rfloor} \cdots \sum_{k_j=0}^{\lfloor \frac{m-2h-\sum_{n=j+1}^{\ell-\frac{1}{2}} (2n+1)k_n}{2j+1} \rfloor} \cdots \sum_{k_1=0}^{\lfloor \frac{m-2h-\sum_{n=\frac{1}{2}}^{\ell-\frac{1}{2}} (2n+1)k_n}{3} \rfloor} \left( m - 2 \left( h + \sum_{j=1}^{\ell-\frac{1}{2}} j k_j \right) \right).$$

While this formula is explicit, it is rather crude. We can, however, adopt standard techniques [57, 58] to write down a generating function for the numbers  $e_m^\ell$ :

$$E^\ell(x) = \sum_{m=0}^{\infty} e_m^\ell x^m = \left( \sum_{i=0}^{\ell-\frac{1}{2}} \frac{x^{2i+1}}{1-x^{2i+1}} \right) \frac{1}{1-x^2} \prod_{j=0}^{\ell-\frac{1}{2}} \frac{1}{1-x^{2j+1}}. \quad (32)$$

We have the following result for the form of the Kac determinant.

**Theorem 13** *The Kac determinant of  $\mathfrak{g}_\ell$  at level  $m$  is given by*

$$\mathcal{D}_m^\ell = C_m^\ell \mu^{e_m^\ell} \prod_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} \left( 2\delta - 2j + \left( \ell + \frac{1}{2} \right)^2 \right)^{d_{m-2(j+1)}^\ell},$$

for some constant  $C_m^\ell$ , and where  $d_{m-2(j+1)}^\ell$  and  $e_m^\ell$  are determined by the generating functions (28) and (32) respectively.

Note that in the case  $\ell = 1/2$ , we have

$$\begin{aligned} e_m^{1/2} &= \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} (m - 2h) = \left( m - \left\lfloor \frac{m}{2} \right\rfloor \right) \left( \left\lfloor \frac{m}{2} \right\rfloor + 1 \right) \\ &= \begin{cases} \frac{1}{4}m(m+2); & m \text{ even} \\ \frac{1}{4}(m+1)^2; & m \text{ odd.} \end{cases} \end{aligned}$$

and from equation (29),

$$d_{m-2(j+1)}^{1/2} = \left\lfloor \frac{m - 2(j+1) + 2}{2} \right\rfloor = \left\lfloor \frac{m - 2j}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor - j.$$

The result of Theorem 13 therefore confirms the conjectured form of the Kac determinant for  $\ell = \frac{1}{2}$  that was presented in [52].

## 5 Concluding remarks

The main results of this paper are presented in Theorem 5 and Theorem 13. Theorem 5 is the culmination of our study of singular vectors in the Verma module of  $\mathfrak{g}_\ell$ , and classifies all the irreducible highest weight modules. Theorem 13 gives the form of the Kac Determinant, that ultimately comes about as a result of the study of singular vectors. For the special case  $\ell = 1/2$ , we have demonstrated explicitly that the conjectured form of the Kac determinant presented in [52] is a special case of the result of Theorem 13.

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